A POSITIVE MASS THEOREM FOR TWO SPATIAL DIMENSIONS

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Abstract. We observe that an analogue of the Positive Mass Theorem in the time-symmetric case for three-space-time-dimensional general relativity follows trivially from the Gauss-Bonnet theorem. In this case we also have that the spatial slice is diffeomorphic to $\mathbb{R}^2$.

In this short note we consider Einstein's equation without cosmological constant, that is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu} \ ,$$

on $1+2$ dimensional space-times. This theory has long been considered as a toy model with possible applications to cosmic strings and domain walls, or to quantum gravity. For a survey please refer to [Bro88, Car98] and references within. This low dimensional theory is generally considered as un-interesting [Col77] due to the fact that Weyl curvature vanishes identically in $(1+2)$ dimensions, a fact often interpreted in the physics literature as the theory lacking gravitational degrees of freedom. Furthermore, the theory does not reduce in a Newtonian limit [BBL86]: the exterior space-time to compact sources is necessarily locally flat and is typically asymptotically conical [Car98, DJtH84, Des85].

For these space-times, by considering point-sources, it is revealed [DJtH84] that the mass should be identified with the angle defects near spatial infinity. For static space-times with spatial sections diffeomorphic to $\mathbb{R}^3$, it is also known that the asymptotic mass can be related to the integral of scalar curvature on the spatial slice, and hence under a dominant energy assumption must be positive.

The purpose of this note is to remark that the topological assumption is unnecessary.

Throughout we shall assume that (Σ, g) is a complete two-dimensional Riemannian manifold which represents a *time-symmetric* spatial slice in a three-dimensional Lorentzian space-time $(M, \bar{g})$ (that is, trace of the second fundamental form of $\Sigma \hookrightarrow M$ vanishes identically; in other words, the slice is *maximal*). We assume that the dominant energy condition holds for $\bar{g}$, and in particular the ambient Einstein tensor satisfies $\bar{G}_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ for any time-like ξ^μ. The Gauss equation then immediately implies that g has non-negative scalar curvature.

Definition 1. A complete two-dimensional Riemannian manifold (Σ, g) is said to be *asymptotically conical* if there exists a compact subset $K \subsetneq \Sigma$ where $\Sigma \setminus K$ has finitely many connected components, and such that if E is a connected component of $\Sigma \setminus K$, there exists a diffeomorphism $\phi : E \to (\mathbb{R}^2 \setminus \bar{B}(0,1))$ where in the Cartesian

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coordinates on $\mathbb{R}^2$ the line element satisfies

$$\mathrm{d}s^2 - \left[\mathrm{d}x^2 + \mathrm{d}y^2 - \frac{1-P^2}{x^2+y^2}\,(x\ \mathrm{d}y - y\ \mathrm{d}x)^2\right] \in O_2\left((x^2+y^2)^{-\epsilon}\right)$$

for some $\epsilon > 0$ and $P > 0$. The notation $f \in O_2(r^{-2\epsilon})$ is a shorthand for

$$|f| + r\,|\partial f| + r^2\left|\partial^2 f\right| \le C r^{-2\epsilon} .$$

Remark 2. The decay condition is sufficient to imply that the scalar curvature S of g is integrable on Σ. Note that in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ the conical metric in the square brackets can be written as the conical

$$\mathrm{d}r^2 + P^2 r^2 \mathrm{d}\theta^2$$

where we see that $m = 2\pi(1-P)$ is the angle defect for parallel transport around the tip of the cone.

Theorem 3. *If (Σ, g) is a complete asymptotically conical two-dimensional orientable Riemannian manifold with pointwise non-negative scalar curvature, then Σ is diffeomorphic to $\mathbb{R}^2$ and $m = 2\pi(1-P)$ is non-negative. If furthermore $m = 0$ then (Σ, g) is isometric to the Euclidean plane.*

Proof. Enumerate from $1 \ldots N$ the asymptotic ends (E_i, g_i) with diffeomorphisms ϕ_i and constant P_i. By the asymptotic structure, for sufficiently large R_i the curve $\gamma_i = \left\{ \phi_i^{-1}(x,y) \;\middle|\; x^2+y^2 = R_i^2 \right\}$ in the end Σ_i has positive geodesic curvature, if we choose the orientation so that the inward normal is toward the compact set K. Let $\Sigma_0 \supsetneq K$ denote the compact manifold with boundary in Σ that is bounded by the γ_i. Applying Gauss-Bonnet theorem, using the fact that the geodesic curvatures are all signed and the scalar curvature is non-negative, we have that Σ_0 has positive Euler characteristic. As Σ_0 is orientable and connected, and has nonempty boundary, it must be diffeomorphic to a disc. Hence Σ has only one asymptotic end and is diffeomorphic to $\mathbb{R}^2$. For sufficiently large R we let Σ_R denote the compact region bounded by $x^2 + y^2 = R^2$. Using the Gauss-Bonnet theorem again, along with the decay properties of the metric, we see that m, the angle defect, is in fact given by

$$m = \lim_{R\to\infty} \frac{1}{2}\int_{\Sigma_R} S\ \mathrm{dvol}_g = \frac{1}{2}\int_{\Sigma} S\ \mathrm{dvol}_g .$$

Hence m is necessarily nonnegative, with equality to 0 only in the case $S \equiv 0$. $\square$

Remark 4. One can analogously define the "quasilocal mass" m_γ associated to $\gamma \subsetneq \Sigma$ a simple closed curve by letting m_γ be the angle defect for parallel transport around γ. Then it is easy to see the this quantity has a monotonicity property: if γ_1 is to the "outside" of γ_2, let $\Sigma_{1,2}$ be the annular region bounded by the two curves, we must have

$$m_{\gamma_1} - m_{\gamma_2} = \frac{1}{2}\int_{\Sigma_{1,2}} S\ \mathrm{dvol}_g .$$

Remark 5. It was pointed out to the author by Julien Cortier that some similar considerations in the asymptotically hyperbolic case was mentioned by Chruściel and Herzlich; see Remark 3.1 in [CH03].

Indeed, Theorem 3 follows also from some more powerful classical theorems in differential geometry. The topological classification can be deduced from, e.g. Proposition 1.1 in [LT91]. One can also deduce the theorem (with some work) from Shiohama's Theorem A [Shi85]. As shown above, however, in the very restricted

case considered in this note the desired result can be obtained with much less machinery.

Remark 6. The author would also like to thank Gary Gibbons for pointing out that a similar argument to the proof of Theorem 3 was already used by Comtet and Gibbons (see end of section 2 of [CG88]) to establish a positive mass condition on cylindrical space-times about a cosmic string; the main difference is that in the above theorem we contemplate, and rule out, the possibility of multiple asymptotic ends, as well as non-trivial topologies inside a compact region.

Remark 7. One can also ask about asymptotically cylindrical spaces, which can be formally viewed as a limit of cones. Indeed, if in Definition 1 we replace the asymptotic condition

$$\mathrm{d}s^2 \to \mathrm{d}r^2 + P^2 r^2 \mathrm{d}\theta^2$$

with

$$\mathrm{d}s^2 \to \mathrm{d}r^2 + (P^2 r^2 + p^2)\mathrm{d}\theta^2 \ ,$$

then the limit $P = 0$ is no longer degenerate, and in fact corresponds to a spatial slice that is cylindrical at the end. In this case, however, the topological statement in Theorem 3 is no longer true: the standard cylinder $\mathbb{S}^1 \times \mathbb{R}^1$ is flat, is asymptotically cylindrical, and has *two* asymptotic ends. However, it is easily checked using the same method of proof as Theorem 3 that this is the only multiple-ended asymptotically cylindrical surface to support a non-negative scalar curvature.

References

[BBL86] John D. Barrow, A. B. Burd, and David Lancaster, *Three-dimensional classical spacetimes*, Classical Quantum Gravity **3** (1986), no. 4, 551–567. MR 851288 (87h:83023)

[Bro88] J. D. Brown, *Lower dimensional gravity*, World Sci. Publishing, 1988.

[Car98] Steven Carlip, *Quantum gravity in 2 + 1 dimensions*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1998. MR 1637718 (99e:83026)

[CG88] A. Comtet and G. W. Gibbons, *Bogomol'nyĭ bounds for cosmic strings*, Nuclear Phys. B **299** (1988), no. 4, 719–733. MR 936758 (89h:83042)

[CH03] Piotr T. Chruściel and Marc Herzlich, *The mass of asymptotically hyperbolic Riemannian manifolds*, Pacific J. Math. **212** (2003), no. 2, 231–264. MR 2038048 (2005d:53052)

[Col77] Peter Collas, *General relativity in two- and three-dimensional space-times*, American Journal of Physics **45** (1977), no. 9, 833.

[Des85] S. Deser, *On the breakdown of asymptotic Poincaré invariance in $D = 3$ Einstein gravity*, Classical Quantum Gravity **2** (1985), no. 4, 489–495. MR 795095 (86k:83020)

[DJtH84] S. Deser, R. Jackiw, and G. 't Hooft, *Three-dimensional Einstein gravity: dynamics of flat space*, Ann. Physics **152** (1984), no. 1, 220–235. MR 734213 (85k:83004)

[LT91] Peter Li and Luen-Fai Tam, *Complete surfaces with finite total curvature*, J. Differential Geom. **33** (1991), no. 1, 139–168. MR 1085138 (92e:53051)

[Shi85] Katsuhiro Shiohama, *Total curvatures and minimal areas of complete surfaces*, Proc. Amer. Math. Soc. **94** (1985), no. 2, 310–316. MR 784184 (86h:53047)

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